Abstract: In passive walking, dissipation due to impacts or damping is offset by the use of potential energy supplied by walking down a slope. In this paper, we develop an analytical procedure to prove the existence and find active limit cycles of a rigid biped in 3-dimensional space. From an existing passive limit cycle, we use the theoretical framework of dynamic geometry and energy shaping, to develop a nonlinear feedback control law which allows the robot to reach stable gaits corresponding to various velocities. Finally, an example that treat a biped robot with knees is presented to illustrate the theoretical results.

Keywords: bipedal gaits, passive walking, limit cycle, energy shaping, nonlinear control

1. INTRODUCTION

In this paper we will focus on “passive dynamic walking”. In fact, the term ‘passive walker’ refers to a mechanism that does not require outside control or actuation to maintain gait. The only source of energy is due to gravity or a torsional spring. This concept was introduced by McGeer (McGeer, 1990a) (McGeer, 1990b), and subsequently studied by several researchers, Collins (Collins et al., 2001), Garcia (Garcia et al., 1998), Goswami (Goswami et al., 1997) (Goswami et al., 1998). They realized that with the appropriate geometry and mass distribution of the walker, and the inclination of the plane, the walker would exhibit stable passive gait.

Moreover, the gaits found typically exist for only shallow slopes and exhibit extreme sensitivity to slope magnitude. Goswami (Goswami et al., 1997) and Spong (Spong, 1999a) (Spong, 1999b) used active feedback control which was based on the passivity property of the biped. They applied a non-linear control to a compass gait biped so that the limit cycle or periodic gait becomes slope invariant.

Kuo (Kuo, 1999) studied 3-dimensional passive walking and found passive gaits in the sagittal and lateral planes, but the lateral motion was unstable.
which was stabilized by feedback control of step width. Three dimensional passive walking has also been obtained by Collins (Collins et al., 2001), where the biped has knees, and arms which swing in coordination with the legs to produce highly anthropomorphic stable gaits. Recently, Spong (Spong and Bullo, 2002) showed that passivity based control can be applied to obtain stable walking for 3-dimensional N-DOF biped robots. It also develops a new active energy based control to increase the basin of attraction of the limit cycle.

Other recent research in biped walking focus on passive velocity field control (PVFC), Asano and Yamakita (Yamakita et al., n.d.) (Yamakita and Asano, 2001a). In another work (Yamakita and Asano, 2001b), the authors they use virtual gravity control for Biped gait synthesis. Grizzle et al. in (Grizzle et al., 2001) and (Westervelt et al., 2003) use switching and PI feedback control to stabilize the biped and to control the speed. A new of using torso to control and stabilize gaits was used by Howell (Howell and Baillieul, 1998).

Motivated by the previous research, in this paper, we investigate the notion of passivity and energy shaping; to develop an analytical approach for finding a stable periodic gaits for N-dimensional rigid biped. From an existing passive limit cycle, we’ll show that there exist a nonlinear feedback control and a set of initial conditions which leads the system to move through another stable limit cycle with different progression velocities.

We organize the paper as follows. In section 2, we present the dynamic and impact model of the biped. The main result theorem and its proof is contained in section 3. The approach is applied to a biped robot with knees in section 4. Finally, section 5 is devoted to conclusions and future work.

2. DYNAMIC MODEL OF 3-D BIPED

The configuration of biped with n links (Figure 1) can often be described as a point of the n-dimensional manifold $Q$ defined by $Q = SO(3) \times T^{n-3}$, where $SO(3)$ is the Rotation Group in $\mathbb{R}^3$ and $T^{n-3}$ (such as $T = [0, 2\pi)$) is the $n-3$-torus. Then every $q \in Q$ is represented by a pair $(R, r)$, where $R$ is the orientation of the first link in the 3-dimensional space and $r$ is the shape of multi-body chain. In the case of planar mechanism, the elements of $SO(2)$ can be represented by scalars then $Q$ may be identified by $T^n$.

![Fig. 1. A 3-D biped](image)

2.1 Lagrangian dynamics

The biped state is an element $(q, \dot{q})$ of the tangent bundle $TQ$, where $q = [q_1, ..., q_n]^T$ is the vector of generalized positions and $\dot{q} = [\dot{q}_1, ..., \dot{q}_n]^T$ is the vector of generalized velocities. The system dynamic is described by Euler-Lagrange equations which are:

$$\frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} = S(q)u$$  \hspace{1cm} (1)

Where $u$ represents the vector of the generalized forces which affect the system.

$S(q)$ is a constant matrix $L : TQ \rightarrow \mathbb{R}$ is the Lagrangian function, such as:

$$L = K - V$$  \hspace{1cm} (2)

with $V$ is the potential energy, and $K$ is the kinetic energy defined by:

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$  \hspace{1cm} (3)

$M(q)$ is the system inertia matrix.

Remark 1. In the case of passive walking ($u = 0$), (1) is identified to the differential equation associated with the langrangian vector field $X_L$.

The expression of the total mechanical energy of the biped gait is:

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + V(q)$$  \hspace{1cm} (4)

where $H : Q \times Q \rightarrow \mathbb{R}$ is the Hamiltonian function and $p = [p_1, ..., p_n]^T$ is the vector of generalized momenta.

Remark 2. The Hamiltonian vector field is defined by:

$$X_H(q, p) = \left( \frac{\partial H(q, p)}{\partial p}, -\frac{\partial H(q, p)}{\partial q} \right)$$  \hspace{1cm} (5)

2.2 Impact dynamic

We assume that the foot-ground contact is inelastic without slipping and the transfer of support.
between swing and stance leg is instantaneous. In the impact moment, the robot configuration does not change, whereas the leg velocities exhibit a jump which results in discontinuity in kinetic energy of the system. The impact dynamic is obtained by integrating the Euler-Lagrange equations (1) over the (infinitesimally small) duration of the impact:

$$\frac{\partial L}{\partial \dot{q}}|_{t^+} = \int_{t^-}^{t^+} F(q, t)dt = W_{t^+}$$

where $W_{t^+}$ represents the impulsive of the contact forces $F$ over the impact.

Furthermore, from (2) and (3) we have:

$$\frac{\partial L}{\partial \dot{q}}|_{t^+} = \frac{\partial K}{\partial \dot{q}}|_{t^+} = M(q)(\dot{q}(t^+)-\dot{q}(t^-))$$

Therefore we close that:

$$W_{t^+} = M(q)(\dot{q}(t^+)-\dot{q}(t^-)) \quad (6)$$

3. MAIN RESULT

3.1 Theorem

We consider n-link biped robot in 3-D space governed by the controlled Euler-Lagrange equations (1) and the impact equation (6). We assume that the biped is fully actuated that is $\text{rang}(S) = n$.

Suppose we are given a vector of initial conditions

$$X = (q(0), \dot{q}(0))^T$$

such as the system (1) with $u = 0$ admit a stable passive limit cycle corresponding to biped gait with the constant progression velocity $v_0$, and $X$ lies in its basin of attraction.

For any desired velocity $v_e$, define $e = \frac{v_e}{v_0}$.

Then with the feedback control law:

$$u = S(q)^{-1}(1 - e^2)\frac{\partial V(q)}{\partial q} \quad (7)$$

where $V$ is the potential energy of the robot system, there is a stable limit cycle corresponding to walking trajectory with the progression velocity $v_e$ and, moreover the vector:

$$X = (q(0), \dot{q}(0))^T$$

lies in its basin of attraction.

3.2 Proof

Since the biped robot considered is an hybrid mechanical system, we will treat separately the simple support phase (continuous dynamic) and the impact event (discrete dynamic). However, let first determine the control law which allow the system mechanical energy shaping.

3.2.1. Energy shaping control

Let $e$ be a strictly positive constant. Motivated by (4), we propose the following form for the desired (closed loop) energy function (or desired Hamiltonian):

$$H_d(q, p) = \frac{1}{2}p^T M_d^{-1}(q)p + V_d(q)$$

Where

$$M_d(q) = \frac{1}{e} M(q) \quad (8)$$

and

$$V_d(q) = eV(q) \quad (9)$$

Then we have

$$H_d(q, p) = e H(q, p) \quad (10)$$

Let $L$ and $L_d$ be the Lagrangians corresponding to Hamiltonians $H$ and $H_d$, respectively. Therefore, the desired system dynamic equation is:

$$\frac{d}{dt} \frac{\partial L_d(q, \dot{q})}{\partial q} - \frac{\partial L_d(q, \dot{q})}{\partial \dot{q}} = 0 \quad (11)$$

When, the closed loop system dynamic is:

$$\frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial q} - \frac{\partial L(q, \dot{q})}{\partial \dot{q}} = S(q)u \quad (12)$$

Since $S$ is invertible (fully actuated system), by replacing (8) and (9) in (11) and equating it with (12), we obtain the expression (7) of the control law which shapes the energy from $H$ to $H_d = e H$.

3.2.2. During the simple support phase:

We present a lemma which will be used in the further developments:

**Lemma 1.** Let $P$ be a (finite dimensional) manifold, $H, K \in F(P)$, and assume that $\Sigma = H^{-1}(h) = K^{-1}(k)$ for $h, k \in \mathbb{R}$ regular values of $H$ and $K$ respectively. Then the integral curves of the vector fields $X_H$ and $X_K$ on the invariant submanifold $\Sigma$ of both $X_H$ and $X_K$ coincide up to a reparametrization.

The proof of this lemma can be found in (Marsden and Ratiu, 1999)

Since the total energy of the biped is continuous during the simple support phase so:

$$H(q, p) \text{ and } H_d(q, p) \text{ is a continuous functions } Q \to \mathbb{R}.$$

Clearly, from (10), $H(q, p) = E_0$ defines the same set as $H_d(q, p) = e E_0$. Let $H^{-1}(E_0) = H_{t^+}^{-1}(e E_0) = \Sigma$. So, to apply lemma 1, it is sufficient to show that $E_0$ and $e E_0$ are a regular values 1 of $H(q, p)$ and $H_d(q, p)$ respectively.

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1 let $P$ a smooth manifold, and $\phi : P \to \mathbb{R}$ a smooth map. A real $r$ is a regular value of $\phi$ if $d\phi(p) = \frac{\partial \phi(p)}{\partial p} \neq 0, \forall p \in \phi^{-1}(r)$
Note that if \((q, p) \in H^{-1}(E_0)\), then \(M^{-1}(q) p > 0\) since \(E_0 > V(q)^2\) for all \(q \in Q\), thus \(dH(q, p) \neq 0\), that is \(E_0\) is a regular value of \(H(q, p)\). Due to (10), we can show by the same way that \(eE_0\) is a regular value of \(H_d(q, p)\).

Therefore we conclude that the integral curves of \(X_H\) and \(X_{H_d}\) on \(\Sigma\) coincide up to reparametrization.

Let \(c(t) = (q(t), p(t))\) an integral curve for \(X_H\) with initial condition \(c_0\) then:
\[
\frac{dc(t)}{dt} = X_H(c(t)), \quad c(0) = c_0
\]
From (5) and (10), we have:
\[
X_{H_d}(q, p) = eX_H(q, p)
\]
Let \(\varepsilon\) be a new parametrization of time \(t\), so, it is easy to show that \(c(\varepsilon)\) is an integral curve for \(X_{H_d}\) with initial condition \(c_0\). That is:
\[
\frac{d(c(\varepsilon))}{d\varepsilon} = X_{H_d}(c(\varepsilon)), \quad c(0) = c_0
\]

Remark 3. we have seen before that if \((q(t), p(t))\) is an integral curve of \(X_H\) (Hamiltonian vector field) then \((q(\varepsilon t), p(\varepsilon t))\) is an integral curve of \(X_{H_d}\), equivalently, if \((q(t), \dot{q}(t))\) is an integral curve of \(X_L\) (Lagrangian vector field) then \((q(\varepsilon t), e \times \dot{q}(\varepsilon t))\) is an integral curve of \(X_{L_d}\).

Therefore, If \(c : t \rightarrow (q(t), \dot{q}(t))\) a periodic trajectory solution of the Euler-Lagrange equations (1), with \((u = 0)\) corresponding to energy \(E_0\) and step period \(T_{\text{step}}\). Then, with a feedback control (7), there exist a periodic trajectory solution of the Euler-Lagrange equations (1) corresponding to energy \(E = e \times E_0\) and step period \(T_{\text{step}}\) defined by \(c_0 : t \rightarrow (q(\varepsilon t), e \times \dot{q}(\varepsilon t))\). Furthermore, since the velocity is inversely proportional to the duration of step, we have:
\[
v_c = e \times v
\]
Where \(v\) and \(v_c\) is the progression velocity, corresponding to passive and active limit cycle, respectively.

3.2.3. During the impact : Let the two trajectories \(c(t) = (q(t), \dot{q}(t))\) and \(c_0(t) = (q_0(t), \dot{q}(t))\) defined as above. We call \(t \neq 0\) the impact time \(c(t)\) and \(c_0(t)\), respectively. In this case from (8), we can easily show that:
\[
M_d(q) \frac{dq_0(\dot{q}(t))}{dt} - M(q) \frac{dq(\dot{q}(t))}{dt} = M(q)(\dot{q}(t^+) - \dot{q}(t^-))
\]
Accordingly, from (6), the impulsive force is invariant under the time parametrization \(t \rightarrow et\),

\[\Delta v_c = \Delta v\]

3.2.4. Conclusion Due to the reasoning above, we can conclude that any limit cycle that exist for the passive walker for one progression velocity \(v\) can be reproduced using by the active control law (7) for any other velocity \(v_c\), also, the two limit cycles coincide under the time reparametrization \(t \rightarrow \frac{et}{c_0}\).

4. SIMULATION EXAMPLE

Knee Strike Model : The knee strike is considered as an event during the swing phase when the thigh and shank of the swing leg have the same angle \((\theta_2 = \theta_3)\) and the swing leg becomes straight. A constraint force is exerted at the knee joint after which it remains locked, i.e., the swing leg remains straight after the knee strike (see Figure 3).

Therefore the swing phase for the biped with knees consists of three phases, the swing leg is bent, knee strike, and the swing leg is straight. The equations of motion are derived by forming the Lagrange Equations. The dynamic equations, from Yamakita ((Yamakita and Asano, 2001a)), are:
\[
M(q)\ddot{q} + C(q; q) \dot{q} + G(q) = -J_T \lambda + B(q)u
\]
where \(q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \); \(B(q) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\)
The constraint of the configuration is given as:

\[ \dot{q}(t) = J_1 q(t) = 0 \]

where \( J_1 \) is the Jacobian matrix. The matrix \( J_1 \) is defined by:

\[ J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and \( u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \) is the control input.

The matrices \( M(q) \), \( C(q; \dot{q}) \), and the vector \( G(q) \) are given as:

\[
M(q) = \begin{bmatrix}
\gamma & \alpha \cos \theta_2 & \beta_1 \cos \theta_3 \\
\alpha \cos \theta_2 & \beta_1 \cos \theta_3 & \beta_2 \cos \theta_3 \\
\beta_1 \cos \theta_3 & \beta_2 \cos \theta_3 & m_2 \beta_3^2 + m_3 \beta_3^2
\end{bmatrix},
\]

\[
C(q, \dot{q}) = \begin{bmatrix}
0 & \alpha \sin \theta_2 \dot{\theta}_2 & \beta_1 \sin \theta_2 \dot{\theta}_3 \\
\alpha \sin \theta_2 \dot{\theta}_2 & 0 & \beta_2 \sin \theta_3 \dot{\theta}_3 \\
-\beta_1 \sin \theta_3 \dot{\theta}_1 & -\beta_2 \sin \theta_3 \dot{\theta}_2 & 0
\end{bmatrix},
\]

\[
G(q) = \begin{bmatrix}
-m_1 a_1 - (m_2 + m_3 + m_H) l_1 g \sin(\theta_1) \\
(m_2 b_2 + m_3 b_2) g \sin(\theta_2) \\
m_3 b_3 g \sin(\theta_3)
\end{bmatrix}
\]

where \( l_1 = a_1 + b_1 \), \( l_2 = a_2 + b_2 \), \( l_3 = a_3 + b_3 \), \( \alpha = -(m_2 b_2 + m_3 b_2) h_1 \), \( \beta_1 = -m_3 b_3 l_2 \), \( \beta_2 = -m_3 b_3 l_2 \), \( \gamma = m_1 a_1^2 (m_H + m_2 + m_3) l_1^2 \) and \( \theta_{ij} = \theta_i - \theta_j \) with \( i, j \in \{1, 2\} \).

The constraint of the configuration is \( \theta_2 = \theta_3 \) differentiating w.r.t time twice, we can write

\[
J_r \ddot{q} = 0 \quad \text{where} \quad J_r = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}
\]

Combining (15) and (14) with \( u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), the constraint force is given by:

\[
\lambda_r = -X_r^{-1} J_r M(q)^{-1} (C(q, \dot{q}) \ddot{q} + G(q))
\]

where

\[
X_r = J_r M(q)^{-1} J_r^T
\]

### 4.2 Collision Phase

As shown in Figure 3, in the collision phase, the biped is essentially a two link biped with both the leg straight. The subscript \( (2) \) denotes coordinate of the two link biped \( q(2) = \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} \).

The impact dynamic equations from Yamakita and Asano (2001a) are:

\[
q_1^+ = J_1 q_2^- \\
Q_1^+ (q_2^-) = Q_1^- (q_2^-)
\]

where the index \( "-" \) means before impact and the index \( "+" \) means after impact. The matrix \( J_1 \), \( Q_1^- (q_2^-) \) and \( Q_1^+ (q_2^-) \) are defined by:

\[
J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

\[
Q_1^- (q_2^-) = \begin{bmatrix} (m_H l_1^2 + 2m_1 a_1 l_1) \cos 2\alpha - \delta a_1 - \delta a_1 \\ -\delta a_1 \end{bmatrix}
\]

\[
Q_1^+ (q_2^-) = \begin{bmatrix} \eta - m_1 l_1 (l_1 - b_1 \cos 2\alpha) \delta b_1 - l_1 \cos 2\alpha \delta b_1 \\ -\delta l_1 \cos 2\alpha \delta b_1 \end{bmatrix}
\]

where \( \alpha = \frac{\theta_2 - \theta_3}{2} \), \( \delta = m_1 l_1 \)

and \( \eta = m_1 a_1^2 - m_H l_1^2 \)

For \( u = 0 \), it was shown by Yamakita et al. (Yamakita and Asano, 2001a) that the biped robot with knees admit a passive limit cycle corresponding to the 3rd slope, the mechanical energy \( E_0 = 156.025J \) and the velocity \( v_0 = 0.734 m/s \). The initial conditions (16) of this limit cycle were determined from the momentum equations using numerical search procedure.

\[
X_0 = \begin{bmatrix} -0.220962 \\ 0.325682 \\ 0.325682 \\ 1.083430 \\ 0.358828 \\ 0.358828 \end{bmatrix}
\]

Then for any velocity \( v_0 \), we apply the approach developed above to obtain the corresponding limit cycle.

It will be defined by the following initial conditions:

\[
X_e = \begin{bmatrix} -0.220962 \\ 0.325682 \\ 0.325682 \\ 1.083430 e \\ 0.358828 e \\ 0.358828 e \end{bmatrix}
\]

such that \( e = \frac{v}{v_0} \).

Motivated by (7) and using (14), we calculate the necessary energy shaping control law:

\[
u = \begin{cases} B^{-1} v & ; \text{If The knee is unlocked} \\ B^{-1} (v - J_1 J_i^{-1} X \times M^{-1} v) & ; \text{If not} \end{cases}
\]
where $v = (1 - e^2) \frac{\partial V(q)}{\partial q}$.

Figure 4 shows stable limit cycles for different velocities.

Fig. 4. Limit Cycles for Various Velocities, $\theta_s$ and $\theta_{ns}$ are respectively the angles of support and non support legs

5. CONCLUSIONS AND FUTURE WORK

Many research has shown that passive walking is an energetic efficient and mechanical cheap way of walking. On the basis of this concept, and as a continuation of the results found in (Spong and Bullo, 2002), we have investigate the energy shaping based control in the biped gait. In fact, from a given passive limit cycle, it has been shown that we are able to obtain active limit cycles for any value of progression velocity. As a simulation example, the approach have been applied to a biped robot with knees moving on inclined plane. However, In practice, there would be limitations to the range of progression velocity achievable due to actuator saturation and ground friction. These additional effects will be a subject of further investigation. Also we hope to include simulations for others forms of biped robot (with torso and knees for example). Furthermore, Our approach may be a used to realize an on line adjustment of the biped gait velocity.

REFERENCES


